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# Calabi-Yau metric on the Fermat surface. Isometries and totally geodesic submanifolds

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**Abstract.** *The hyper – Kahler Calabi-Yau metric* $m$ *on the Fermat surface*

$$F = (x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0) \subset \mathbb{C}P^3$$

*associated with the embedding* $F \subset \mathbb{C}P^3$ *is studied. We prove that the lattice of integer parallel* 2*-forms on the Riemannian manifold* $(F, m)$ *has the Gram matrix* $\mathrm{diag}(4, 8, 8)$. *We use it for calculation of the isometry group* $\mathrm{Isom}(m)$. *The action of this group on the twistor space of parallel complex structure on* $(F, m)$ *is described and the existence of 10 complex structures with non-trivial stabilizer in* $\mathrm{Isom}(m)$ *is established. Then we give the classification of all connected* 2*-dimensional totally geodesic submanifolds which are fixed points sets of isometries. There are 288 such manifolds of genus* $0, 1, 2, 3, 5$. *They are complex curves respect to one of the 5 (up to a sign) distinguished complex structures.*

*From the physical point of view such submanifolds are interpreted as (holomorphic) instantons for sigma model with the value in* $K3$ *surface. Such instantons are studied by physicistis in relation with string theory.(*)*

*The generalization of the results to some class of Calabi-Yau metrics on* $K3$ *surfaces* $X$, *associated with the embeddings* $X \subset CP$ *is given.*

*Key-Words: Calabi-Yau,* $K3$ *surface, isometry group.*
*1980 MSC: 53 C 25, 53 C 80.*

(*) See for example: Y. Kogan, D. Markushevich, A. Morozov, M. Olshanetsky, A. Perelomov, A. Rosly, «Some examples of instantons in sigma models. $K3$ manifolds.» Preprint n. 32, 1988, 1-28.

## 1. INTRODUCTION

The Fermat surface $F$ is the complex surface in $\mathbb{C}P^3$ defined by the equation

$$f \equiv x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,$$

where $x_0, \dots, x_3$ are the homogeneous coordinates in $\mathbb{C}P^3$.

The Fermat surface is a $K3$-surface, i.e. a simply-connected compact complex surface with Chern class $c_1 = 0$. Any $K3$-surface as a real manifold is diffeomorphic to $F$ [2].

By Yau theorem [10], any Kähler metric $\kappa$ on a $K3$-surface $X$ determines a Ricci-flat Kähler metric. It is defined as the unique Ricci-flat Kähler metric $m$ whose Kähler form is cohomologous to the Kähler form of the metric $\kappa$. This metric $m$ has two properties:

1) $m$ is hyper-Kähler, i.e. its holonomy group

$$\mathrm{Hol}(m) = Sp(1) = SU(2);$$

2) $m$ is anti-auto-dual, i.e. $*R = -R$, where $R$ is the curvature 2-forms of $m$ and $*$ is the Hodge operator.

Note that for a Riemannian metric on a 4-manifold the properties 1) and 2) are equivalente. By Hitchin theorem [10]. Ricci-flat metrics on $K3$-surfaces exhaust all non-flat hyper-Kähler metrics on compact 4-manifolds.

Any holomorphic embedding $X \overset{\mu}{\hookrightarrow} \mathbb{C}P^N$ of $K3$-surface $X$ defines Kähler metric $\kappa$ on $X$ induced by the Fubini-Study metric on $\mathbb{C}P^N$. The corresponding Ricci-flat metric $m$ on $X$ is called Calabi-Yau metric associated with the embedding. In the paper we study the Calabi-Yau metric $m$ on the Fermat surface $F$ associated with the tautological embedding $F \subset \mathbb{C}P^3$. We shall call such metric standard. The Riemannian manifold $(F, m)$ will be called Fermat manifold. The explicit form of the metric $m$ is not known. Nevertheless, using some results from the geometry of $K3$-surface, we compute the group $I(m)$ of isometries and determine all totally geodesic submanifolds of the Fermat manifold which are fixed points sets of isometries.

In § 2 we establish some properties of the group $I(m)$ of isometries of a hyper-Kähler metric $m$ on a 4-manifold $X$. We study the action of $I(m)$ on the space $E \simeq \mathbb{R}^3$ of parallel 2-forms on $X$ and on the Grassmanian $G_2(E) \simeq \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ of oriented 2-subspace of $E$ which identifies with the manifold of parallel complex structure on $X$. We investigate the fixed points set $\mathrm{Fix}(g)$ of an isometry $g \in I(m)$. Finally, we prove that iff $m$ is the Calabi-Yau metric associated with a linear normal embedding $X \subset \mathbb{C}P^N$ then the group $I(m, \pm J_0)$ of isometries of $m$ which preserve the induced complex structure $J_0$ up to a signe is identified with the group $\mathrm{Aut}(X)$ of (anti) holomorphic transformations of $\mathbb{C}P^N$ preserving $X$. In §3 we determine the lattice $\Gamma \subset E$ of integer parallel 2-forms on the Fermat manifold $(F, m)$. It has Gram matrix $\mathrm{diag}(4, 8, 8)$ with respect to some basis $\epsilon_0 = m \circ J_0, \epsilon_1, \epsilon_2$. Using this result, we calculate in § 4 the group $I(m)$ of isometries explicitely:

$I(m) = G \cup \tau G$, where $\tau$ is the antiholomorphic involution of $F \subset \mathbb{C}P^3$ defined by the complex conjugation of the homogeneous coordinates in $\mathbb{C}P^3$ and $G = I(m, J_0) = S_4 \cdot (\mathbb{Z}_4)^3$ is the group of $J_0$-holomorphic transformations of $F$ generated by permutations of the coordinates $x_0, \dots, x_3$ in $\mathbb{C}P^3$ and its multiplications by $\sqrt[4]{1}$. The group $I(m)$ has the order $2^{10} \cdot 3$ and acts in $E$ as the full group of orientation preserving automorphism of the lattice $\Gamma$, that is the dihedral group $D_4$. In the Grassmannian $G_2(E) = \mathbb{C} \cup \{\infty\}$ of parallel complex structure $J_t, t \in \mathbb{C} \cup \{\infty\}$ there are exactly 10 distinguished complex structures with non-trivial stabilizer in $D_4$ :

1) $J_0, J_\infty = -J_0$;
2) $J_t, t^4 = 1, J_{-t} = -J_t$
3) $J_t, t^4 = -1, J_{-t} = -J_t$.

As indicated above, they decompose into 3 orbits of the group $D_4$. The complex structure $J_0$ is the natural structure of the complex surface $F \subset \mathbb{C}P^3$.

In § 5 we study and enumerate all projective lines on the Fermat surface $F$. There are exactly 48 such lines. They are mutually non-homologous and generate 20-dimensional subspace in $H_2(F, \mathbb{C})$ dual to the space $H^{1,1}(F, \mathbb{C})$. The lines are connected components of the fixed point set of involutions $h \in I(m)$. Hence, they are totally geodesic submanifolds of the Fermat manifolds $(F, m)$.

In § 6 we define some subgroup $B \approx (\mathbb{Z}_2)^5$ of the group $I(m)$ of isometries of the Fermat manifold. It contains $J_0$-antiholomorphic involution $\tau$ and is generated by 6 antiholomorphic involutions $\rho_j, j = 1, \dots, 6$. We describe the fixed points set $\mathrm{Fix}(h)$ for $h \in B$. In particular, we prove that $C_j = \mathrm{Fix}\, \rho_j$ $(j = 1, \dots, 6)$ is topological sphere with 5 handles. Then we show that the subspace $H_2(F, \mathbb{C})^B$ of $B$-invariant elements from $H_2(F, \mathbb{C})$ is 1-dimensional. This implies the existence a unique (up to a signe) $B$-invariant parallel complex structure $J$. Since $C_j (j = 1, \dots, 6)$ is a $B$-invariant $J$-complex curve, the dual harmonic integer 2-form $\gamma_j \in \Gamma$ is $B$-invariant. Hence, it is proportional to $m \circ J$. Since

$$\gamma_j \cdot \gamma_j = C_j \cdot C_j = -\chi(C_j) = 8$$

and the only elements $\gamma \in \Gamma$ with $\gamma \cdot \gamma = 8$ are $\pm\epsilon_1, \pm\epsilon_2$, we can indetify $\gamma_1 = \gamma_2 = \dots = \gamma_6$ with $\epsilon_2$. Then $J$ is identified with the distinguished complex structure $J_1$. Hence $J_1$ is characterized as the unique (up to a signe) $B$-invariant parallel complex structure.

In § 7 we classify all connected 2-dimensional totally geodesic submanifolds $C$ in the Fermat manifold $(F, m)$ which are connected components of fixed points sets $\mathrm{Fix}(g)$, $g \in I(m)$.

They are connected components of the fixed points set $\mathrm{Fix}(h)$ of involutions $h \in I(m)$. There are $288 = 2^5 \cdot 3^2$ such submanifolds with the Euler characteristic $2, 0, -2, -2^2, -2^3$. Thy are complex submanifolds with respect to one of the distinguished complex structures. In particular, the $J_0$-complex submanifolds $C$ are either projective lines (i.e. 48 spheres) or spheres with 3 handles (28); the $J_1$-complex submanifolds are spheres (48), tori (12) or spheres with 5 handles (6); the $J_\chi$-complex submanifolds $(\chi = (1+i)/\sqrt{2}, \chi^4 = -1)$ are spheres (16) or spheres with 2 handles (24).

In § 8 we represent the Fermat surface $F$ with the complex structure $J_1$ by a complete intersection of 3 diagonal quadratic in $\mathbb{C}P^5$ and state that the Calabi-Yau metric $m'$ associated with the embedding $F \subset \mathbb{C}P^5$ is isometric to $\sqrt{2}m$, where $m$ is the standard Calabi-Yau metric on $F$. In this model the group $B$ acts as the projective group $\{\mathrm{diag}(1, \epsilon_1, \dots, \epsilon_5), \epsilon_i^2 = 1\}$.

To construct the embedding $j : F \to j(F) = K \subset \mathbb{C}P^5$ we prove that $J_1$-complex curves of genus 5 $C_j = \mathrm{Fix}\,\rho_j$ $(j = 1, \dots, 6)$ are linear equivalent and define the very ample linear bundle $L = [C_j]$ with $H^0(F, O(L)) \simeq \mathbb{C}^6$. The map $j$ is defined as the map $F \to P(H^0(F, O(L))^*)$ associated with the bundle $L$.

REMARK. 1) Many results of § 6 – 8 remain true if one changes $F$ by an $B$-invariant quartic $X \subset \mathbb{C}P^3$, see § 6 – 8.

2) The explicit construction of the Calabi-Yau metric $m$ on $F$ reduces to the identification of two holomorphically non-equivalent $K3$-surface : $F \subset \mathbb{C}P^3$ and $K \subset \mathbb{C}P^5$, that is, to the explicit description of the diffeomorphism $j : F \to j(F) = K \subset \mathbb{C}P^5$. This diffeomorphism is $I(m, J_1)$-equivariant and it is an isometry of the Riemannian manifolds $(F, m)$ and $(K, m'/\sqrt{2})$.

3) The surface $K$ is a Kummer surface in $\mathbb{C}P^5$ associated with the Picard group $\mathrm{Pic}_0(C)$ of some curve of genus 2. More precisely, $C$ is the 2-fold covering of the Riemannian sphere ramified in 6 vertices of the octahedron. The embedding $K \subset \mathbb{C}P^5$ is constructed as in [3]. Ch. 6.

## 2. PROPERTIES OF ISOMETRIES OF A 4-DIMENSIONAL HYPER-KÄHLER MANIFOLD

Let $m$ be a non-flat hyper-Kähler metric (i.e. a Riemannian metric with the holonomy group $\mathrm{Hol}(m) = Sp(1)$) on a compact 4-manifold $X$. We denote by $E \approx \mathbb{R}^3$ the space of parallel 2-forms on $(X, m)$ with the natural orientation and the Euclidean metric

$$\alpha \cdot \beta = \int_X \alpha \wedge \beta.$$

The manifold of parallel complex structures on $(X, m)$ is identified with the Grassmanian $G_2(E)$ of oriented 2-subspace in $E$ : Parallel complex structure $J$ defines the subspace $\langle J\rangle \subset E$ withe the base $\mathrm{Re}\,\omega, \mathrm{Im}\,\omega$, where $\omega \neq 0$ is a holomorphic 2-form on the complex manifold $(X, J)$. In other words, $\langle J\rangle$ is the orthogonal complement to the Kähler form $m \circ J$ in $E$. The manifold $G_2(E)$ is identified with the complex quadric

$$Q = \{\alpha \in P(\mathbb{C} \otimes E) \simeq \mathbb{C}P^2, \alpha \cdot \alpha = 0\} \simeq \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}.$$

A quantity $t \in \mathbb{C} \cup \{\infty\}$ corresponds to the point $\alpha \in Q$, $\alpha = (2t, i(t^2+1), t^2 - 1)$ and to the 2-subspace of $E^3$ with the base $\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha$. The associated complex structure on $X$ will be denoted by $J_t$. We denote by $\Gamma$ the lattice of integer 2-forms in $E$. Let $I(m)$ be the (finite) group of isometries of the metric $m$ and $I^0(m)$ is the kernel of its natural representation in $E$.

LEMMA 2.1. *Let $m$ be a hyper-Kähler metric on 4-manifold $X$ (It is not supposed that $X$ is compact and simply connected).*

(i) *Any isometry $g \in I(m)$ induces in $E$ a proper orthogonal transformation. Hence, there is a fixed vector $e \neq 0$ for $g$. It is unique (up to a proportionality) if $g \notin I^0(m)$ and it acts in the 2-plane $e^{\perp}$ as the rotation by an angle $\varphi = \varphi(g)$.*

(ii) *For $g \in I(m)$, there is a $g$-invariant parallel complex structure $J^g$ in $X$ defined by the condition $\langle J^g\rangle = e^{\perp}$. It is unique (up to a signe) if $g \notin I^0(m)$.*

(iii) *Let $x$ be a fixed point of $g \in I(m)$. In suitable $J_g$-holomorphic coordinates, the differential $dg_x$ is given by* $\mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}), \varphi_1 + \varphi_2 = \varphi(g)$.

(iv) *For $v \neq 0, v \in T_xX$, let $I_v$ be the group of isometries which preserve $x \in X$ and $v$. Then the map $v \lrcorner : E \to v^{\perp} \subset T_x^*X$, defined by the internal multiplication by $v$, is an isomorphism of $I_v$-modules. In particular, $I_v \cap I^0(m) = \{\mathrm{id}\}$ and for $h \in I_v$, the condition $h^n \in I^0(m)$ implies $h^n = \mathrm{id}$.*

The proof of the lemma is straight forward. ∎

PROPOSITION 2.1. *Let $(X, m)$ be a hyper-Kähler 4-manifold.*

1) *For any isometry $g \in I^0(m), g \neq \mathrm{id}$, the fixed point set $\mathrm{Fix}(g)$ is discrete.*

2) *Distinct involutions from $I^0(m)$ have not common fixed points.*

3) *Suppose that each element from the quotient group $I(m)/I^0(m)$ has even order. Then each 2-dimensional component of the fixed point set $\mathrm{Fix}(g), g \in I(m)$, is contained into the fixed point set $\mathrm{Fix}(h)$ of the involution $h = g^k \in I(m) - I^0(m)$ where $2k = \mathrm{order}\, g$.*

4) *For an involution $h \in I(m) - I^0(m)$, the set $\mathrm{Fix}(h)$ is either empty or has only 2-dimensional components. For distinct involutions $h, h' \in I(m)$ the sets $\mathrm{Fix}(h), \mathrm{Fix}(h')$ have no common connected components.*

5) *Let $h_1, h_2, h_3 \in I(m) - I^0(m)$ be distinct commuting involutions which preserve the same parallel complex structure. Then*

$$\mathrm{Fix}(h_1) \cap \mathrm{Fix}(h_2) \cap \mathrm{Fix}(h_3) = \emptyset.$$

**Table 1.**

| $d(h)$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\lvert \mathrm{Fix}(h) \rvert$ | 8 | 6 | 4 | 4 | 2 | 3 | 2 |

6) *Suppose $X$ is diffeomorphic to the $K3$-surface and* $\mathrm{id} \neq h \in I^0(m)$. *If the order $h = 2, 3$ or $4$, then* $|\mathrm{Fix}(h)| = 8, 6$ *or* $4$ *respectively.*

REMARK. It can be proved that for any $h \in I^0(m)$, the number $|\mathrm{Fix}(h)| \leq 8$ and it is defined by the order $d(h) \leq 8$ of $h$ as it is indicates in the table 1:

*Proof.* The statements 1), 3) follows from iv) and 2), 4) follows from (ii), (iii). The assertion 2) implies 5), since the action of the involutions $h_1, h_2, h_3$ on the space $E$ is identical and, hence, $h_1h_2, h_1h_3 \in I^0(m)$. Now we prove 6). Since $h \in I^0(m)$ preserves the complex structure $J_0$, the $J_0$-holomorphic Lefschetz number [3] $L(h, O)$ is defined. For $K3$-surface, we have $H^{1,0} = 0, H^{0,0} = H^{2,0} \simeq \mathbb{C}$ with the trivial action of $h$. Hence $L(h, O) = \sum tr(h^*|H^{0,i}) = 2$. The holomorphic Lefschetz formula and 1) imply that $\mathrm{Fix}(h)$ is finire and isn't empty. Suppose that order $d(h) = 2, 3$ or $4$. Then, by lemma 2.1. (ii) the differential $dh_x$ of $h$ at any $h$-invariant point $x$ is given by $dh_x = \mathrm{diag}(-1, -1), \mathrm{diag}(e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}})$ or, respectively, $\mathrm{diag}(i, -i)$. The Lefschetz formula is read as:

$$2 = L(h, O) = \sum_{x \in \mathrm{Fix}(h)} \frac{1}{\det(\mathrm{id} - dh_x)} = \frac{N}{4}, \frac{N}{3} \text{ or } \frac{N}{2}.$$

This proves 6). ∎

Now we assume that $X$ is an algebraic $K3$-surface with complex structure $J_0$ and $m$ is the Calabi-Yau metric on $X$ associated with a linear normal embedding

$$\mu : X \to \mathbb{C}P^N = P(H^0(X, O(L))^*)$$

where $L$ is a very ample line bundle [3].

PROPOSITION 2.2. *The group $I(m, \pm J_0)$ of isometries of the Calabi-Yau metric $m$ which preserve the complex structure $J_0$ up to a signe is identified with the group* $\mathrm{Aut}(\mu(X))$ *of holomorphic and anti-holomorphic transformations of $\mathbb{C}P^N$ which preserve $\mu(X)$ :*

$$I(m, \pm J_0) = \mathrm{Aut}(\mu(X))|_{\mu(X)}.$$

*Proof.* Denote by $\eta$ the Kähler form of the Fubini-Study metric $\kappa$ on $\mathbb{C}P^N$. The cohomology class $[\eta]$ is dual to the homology class of hyperplane section. Let $g$ be a holomorphic (hence, projective) transformation of $\mathbb{C}P^N$ preserved $\mu(X)$. Then it preserves the cohomology class $[\eta|\mu(X)]$. Hence, by Yau theorem, it preserves the Calabi-Yau metric $m$ associated with the embedding as the unique Ricci-flat Kähler metric $m$ with Kähler form $m \circ J_0 \in [\eta|\mu(X)]$. The similar argument applies to the case when $g$ is antiholomorphic transformation and, hence, $g_* J_0 = -J_0, [g^*\eta] = -[\eta]$.

Now we prove the converse assertion. Note that the line bundle $L$ is the restriction on $\mu(X)$ of the hyperplane bundle $[H]$ over $\mathbb{C}P^N$ [3]. Hence, the Chern class $c_1(L)$ is equal to $[\eta|\mu(X)]$. Since $X$ is simply connected, there is only one (up to an isomorphism) line bundle with the Chern class $[\eta|\mu(X)]$. Let $g \in I(m, \pm J_0)$. For clarity, we assume that $g$ preserves the complex structure $J_0$. Then $g$ preserves the Kähler form $m \circ J_0$ and, hence, the cohomology class $[m \circ J_0] = c_1(L)$. This implies that the line bundle $g^*L$ is isomorphic to $L$. Hence, $g$ is covered by an automorphism $\hat{g}$ of the bundle $L$. Since any fiber preserving automorphism of $L$ is the multiplication by a constant, the automorphism $\hat{g}$ defines up to a constant factor preserves the projective embedding $\mu : X \to P(H^0(X, O(L))^*) = \mathbb{C}P^N$. This proves the proposition.

■

## 3. THE LATTICE $\Gamma$ OF INTEGER PARALLEL 2-FORMS ON THE FERMAT SURFACE

Let $(F, m)$ be the Fermat manifold. We shall denote by $J_0$ the standard complex structure on $F$ and by $\epsilon_0 = m \circ J_0$ the Kähler form of the metric $m$. To calculate the isometry group $I(m)$ we need the explicit description of the lattice $\Gamma$ of integer parallel 2-forms on the hyper-Kähler manifold $(F, m)$. It derives from the following fundamental for us result of I.I. Piatetskii-Shapiro and I.R. Shafarevitch.

THEOREM 3.1. ([1], § 8). *Let $S \subset H_2(F, \mathbb{R})$ be the lattice of complex algebraic cycles on the Fermat surface $F$ and $T \subset H^2(F, \mathbb{Z})$ is the lattice of all integer cohomology classes annulated by cycles from $S$.*

1) *The lattice $T$ has a basis $\epsilon_1, \epsilon_2$ with the scalar products $\epsilon_1 \cdot \epsilon_2 = 0$, $\epsilon_1^2 = \epsilon_2^2 = 8$. In particular, $F$ is a singular Kummer surface.*

2) *Discriminante of the lattice $S$ is equal to 64.* ■

Using this result, we prove

THEOREM 3.2. *The lattice $\Gamma$ of integer parallel 2-forms on the hyper-Kähler manifold $(F, m)$ has an orthogonal basis $\epsilon_0, \epsilon_1, \epsilon_2$ with $\epsilon_0^2 = 4, \epsilon_1^2 = \epsilon_2^2 = 8$. In particular, for any $\gamma \in \Gamma, \gamma^2 \equiv 0 (\mathrm{mod}\ 4)$.*

*Proof.* The lattice $T$ is naturally identified with the sublattice $T'$ of the lattice $\Gamma$ which consists of parallel 2-form orthogonal to the Kähler form $\epsilon_0$. Therefore $\Gamma$ contains the sublattice $\Gamma' = \mathbf{Z}\epsilon_0 \oplus T'$ with the orthogonal basis $\epsilon_0, \epsilon_1, \epsilon_2$, where $\epsilon_0^2 = \deg F = 4, \epsilon_1^2 = \epsilon_2^2 = 8$.

We need only to prove that $\Gamma = \Gamma'$. This follows from the facts:

1) $T' = \Gamma \cap (T' \otimes \mathbf{Q})$ (theorem 3.1)

2) the lattice $\Gamma$ is even and integer as a sublattice of the even lattice $H^2(F, \mathbf{Z}) \simeq 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8)$ [2].

Indeed, if we suppose that $\Gamma \neq \Gamma'$ then there is an element $\gamma \in \Gamma$ of the form $\gamma = \alpha_0\epsilon_0 + \alpha_1\epsilon_1 + \alpha_2\epsilon_2$, $0 < \alpha_0 < 1$, $0 \leq \alpha_i < 1$, $i = 1,2$. The conditions $\gamma^2 \in 2\mathbf{Z}, \gamma \cdot \epsilon_j \in \mathbf{Z}, j = 0,1,2$ imply $4\alpha_0 \in \mathbf{Z}$, $8\alpha_i \in \mathbf{Z}, i = 1,2$, and elementary arithmetic arguments lead to the contradiction. ∎

## 4. COMPUTATION OF THE GROUP OF ISOMETRIES

The result of § 3 imply the following.

THEOREM 4.1. *Let $(F, m)$ be the Fermat manifold. Then the group of isometries $I(m)$ is identified with the group of all holomorphic and antiholomorphic transformation of $\mathbf{C}P^3$ which preserved the surface $F$.*

*Proof.* An isometry $g \in I(m)$ preserves the lattice $\Gamma$. Hence, it preserves the pair of vectors $\pm\epsilon_0 \in \Gamma$ as the only vectors with the scalar square 4. This shows that $g_* J_0 = \pm J_0$. Now the theorem follows immediately from the proposition 2.2. ∎

Denote by $\tau$ the anti-holomorphic transformation of $\mathbf{C}P^3$ (and also its restriction on $F$) which is induced by the complex conjugation of the homogeneous coordinates. By theorem 4.1., $I(m) = \mathrm{Aut}^+(F) \cup \tau \circ \mathrm{Aut}^+(F) = \mathrm{Aut}(F)$. Hence, the computation of the isometry group $I(m)$ reduces to the computation of the group $\mathrm{Aut}^+(F)$ of all projective transformations of $\mathbf{C}P^3$ which preserve $F$. We prove more general result.

Denote by $F_{d,n}$ the Fermat hypersurface in $\mathbf{C}P^{n-1}$ defined by

$$x_0^d + \ldots + x_{n-1}^d = 0.$$

PROPOSITION 4.1. The group $\mathrm{Aut}^+(F_{d,n})$ of all projective transformations of $\mathbf{C}P^{n-1}$ which preserve $F_{d,n}$ is the semi-direct product $\mathrm{Aut}^+(F_{d,n}) = (\mathbf{Z}_d)^{n-1} \cdot S_n$ of the group $S_n$ of all permutations of the coordinates and the group $(\mathbf{Z}_d)^{n-1} = \{\mathrm{diag}(1, \epsilon_1, \ldots, \epsilon_{n-1}), \epsilon_i^n = 1\}$ (We describe a projective transformation $g$ by the matrix $A$ of the corresponding linear transformation $\tilde{g}$ of $\mathbf{C}^n$ defined up to a scale factor).

We prove the proposition by the induction on $n$. For $n = 2$, the proof is straight forward. Let now $n > 2$ and $g \in \mathrm{Aut}^+(F_{d,n}) = G$. Denote by $A = (A_{ij})$ the matrix which represents $g$ and preserves the function $f = x_0^d + \ldots + x_{n-1}^d$. Then

$$\sum u_i^d = \sum x_i^d, \tag{4.1}$$

where $u_i = \sum x_j A_{ji}$.

Multiplying $g$ by some element from $G$, we may assume that $A_{n-1,n-1} \neq 1$ and $u_{n-1} \notin \mathbb{C} x_{n-1}$. These imply that

(i) the linear functions $x_0, \ldots, x_{n-2}$ on $\mathbb{C}^n$ are linear independent modulo $v = u_{n-1} - x_{n-1}$ and

(ii) $x_{n-1} \not\equiv 0 (\mathrm{mod}\ v)$.

The identity (4.1) implies

$$\sum_{i \leq n-2} u_i^d - \sum_{i \leq n-2} x_i^d = x_{n-1}^d - u_{n-1}^d \equiv 0 (\mathrm{mod}\ v).$$

By (i), we may consider $x_0, \ldots, x_{n-2}$ as the coordinates on the subspace $v = 0$. Using inductive assumption, we can write

$$u_i \equiv \epsilon_{s(i)} x_{s(i)} (\mathrm{mod}\ v),$$

where $s$ is some permutation and $\epsilon_j^d = 1$. Multiplying $g$ by some element from $G$, we may assume that

$$u_i = x_i + \lambda_i v, \qquad i \leq n-2; \qquad u_{n-1} = x_{n-1} + v.$$

Then we have the identity

$$0 \equiv \frac{1}{vd} \left[ \sum_{i \leq n-1} u_i^d - \sum_{i \leq n-1} x_i^d \right] \equiv x_{n-1}^{d-1} + \sum_{i \leq n-2} \lambda_i x_i^{d-1} \quad (\mathrm{mod}\ v).$$

The conditions (i), (ii) imply that all $\lambda_i$ except one vanish. This reduce the assertion to the case $n = 2$. ∎

Returning to the Fermat surface $F = F_{4,4}$, we fixed the following notations: $i_p = \mathrm{diag}(\underset{1}{1}, \underset{2}{1}, \ldots, \underset{p1}{1}, \underset{p}{i}, 1, \ldots, 1)$, $i = \sqrt{-1}, \sigma_p = i_p^2, \alpha_{pq}$ is the transposition of the homogeneous coordinates $x_p$ and $x_q (p, q = 0, \ldots, 3)$.

COROLLARY 4.1. *Let* $(F, m)$ *be the Fermat manifold. Then* $I(m) = G \cap \tau G$, *where* $G = (\mathbb{Z}_4)^3 \cdot S_4$ *is the group of all holomorphic transformations of* $F$ *generated by the transformations* $i_p, \alpha_{pq} (p, q = 0, \ldots, 3)$. ∎

We notice that any projective transformation $g$ which preserve $F$ can be represent by an linear transformation $\tilde{g}$ of $\mathbb{C}^4$ which preserves the function $f = x_0^4 + \ldots x_3^4$. The transformation $\tilde{g}$ is defined up to a factor $\epsilon \in \mathbb{C}, \epsilon^4 = 1$. Hence, we have the homomorphism

$$\det : G \to \mathbb{C}^*, \quad g \mapsto |g| = \det \tilde{g}.$$

Corollary 4.1. implies that $\det(G) = \mathbb{Z}_4 = \{\epsilon, \epsilon^4 = 1\}$.

We set

$$G^0 = \{g \in G, |g| = 1\}, \quad H = \{g \in G, |g| = \pm 1\}.$$

We have a chain of the normal subgroups

$$G^0 \subset H \subset G \subset I(m).$$

LEMMA 4.1. *Let* $\omega \neq 0$ *be a holomorphic* 2*-form on* $F$. *Then the action of* $g \in G$ *on* $\omega$ *is given by*

$$g^* \omega = |g| \omega.$$

In particular, $G^0 = I^0(m)$.

*Proof.* To construct $\omega$ explicitely, we define the holomorphic 3-form $\gamma$ of Gelfand-Lerey on the surface $S = \{f \equiv x_0^4 + \ldots + x_3^4 = 0\} \subset \mathbb{C}^4$ by the relation

$$df \wedge \gamma = d^4 x.$$

Contracting $\gamma$ with the radial vector fields $E = \sum x_i \frac{\partial}{\partial x_i}$ we receive 2-form $\tilde{\omega} = E \rfloor \gamma$ on $S$. It is $E$-invariant $(E \cdot \tilde{\omega} = 0)$ and $E$-horizontal $(E \rfloor \tilde{\omega} = 0)$. Hence, it defines a holomorphic 2-form $\omega$ on $F = PS$. For $g \in G$, we have $\tilde{g}^*(d^4 x) = |g| d^4 x$, $\tilde{g}^* f = f$, $\tilde{g}_* E = E$. These imply the lemma. ∎

COROLLARY 4.2.

1) *The group* $\frac{I(m)}{I^0(m)} = \frac{G \cup \tau G}{G^0}$ *is isomorphic to the group of all orientation preserving automorphisms of the lattice* $\Gamma$, *that is to the dihedral group* $D_4$. *All elements from* $\frac{I(m)}{I^0(m)} = D_4$ *have even orders.*

2) *There are exactly 10 points on the Riemannian sphere* $\mathbb{C}P^1$ *of the parallel complex structures* $J_t$ *(see § 1) which have non-trivial stabilizer in the group* $\frac{I(m)}{I^0(m)}$. *They are decomposed into 3 orbits under* $\frac{I(m)}{I^0(m)}$ :

1) $t = 0, \infty$ 2) $t, t^4 = 1$, 3) $t, t^4 = -1$.

*The corresponding subspace of* $E$ *is generated by the following lattice:*

1) $\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ *for* $t = 0$, 2) $\mathbb{Z}\epsilon_0 \oplus \mathbb{Z}\epsilon_1$ *for* $t = 1$, 3) $\mathbb{Z}\epsilon_0 \oplus \mathbb{Z}(\epsilon_1 + \epsilon_2)$ *for* $t = \chi := \frac{i+1}{\sqrt{2}}$.

*Moreover,* $J_\infty = -J_0, J_{-t} = -J_t$ *for* $t, t^4 = \pm 1$. ■

The corollary 4.2. 1) and the proposition 2.1. 3) imply that the determination of 2-dimensional fixed point set $\mathrm{Fix}(g)$, $g \in I(m)$ reduces to the case when $g$ is an involution.

In the next paragraph we shall determine such sets which are projective lines.

## 5. PROJECTIVE LINES ON THE FERMAT SURFACE

We determine all projective lines on the Fermat surface $F$ and prove that they are totally geodesic submanifolds with respect to the Calabi-Yau metric $m$. Let $\Delta = \{(x_0 : x_1 : x_2 : x_3), x_0x_1x_2x_3 = 0\}$ be the coordinate tetrahedron in $\mathbb{C}P^3$. Let $\Delta_i' = \{x_0 = x_i = 0\}$ and $\Delta_i (i = 1, 2, 3)$ are the pair of its opposite edges.

The intersection of each edge $\Delta_i$ (respectively $\Delta_i'$) with the generalized Fermat surface $F^d = \{x_0^d + \ldots + x_3^d = 0\}$ of the degree $d$ consists of the $d$ points $P(i, a)$ (respectively $P'(i, a)$) with the homogeneous coordinates

$$(x_0 : x_i : x_j : x_k) = \begin{cases} (\chi : \epsilon^a : 0 : 0) & \text{for } P(i, a) \\ (0 : 0 : \epsilon^b : \chi) & \text{for } P'(i, b). \end{cases}$$

Here $(i, j, k)$ is a cyclic permutation of the indices $(1, 2, 3), \chi^2 = \epsilon$, $\epsilon = \sqrt[d]{1}$ is a fixed primitive root of the degree $d$ from the unit.

Denote by $l(i, a, b)$ the projective line which passes trough the points $P(i, a)$ and $P'(i, b)$. Obviously, $l(i, a, b) \subset F^d$.

PROPOSITION 5.1.

1) *All projective lines on the Fermat surface* $F^d \subset \mathbb{C}P^3$ *are exhausted by the lines* $l(i, a, b), (i, a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_d \times \mathbb{Z}_d, d > 2$.

2) *Two distinct lines* $l(i, a, b)$, $l(i', a', b')$ *intersect iff one of the following conditions is satisfied:*

1) $i = i' - 1$, $a + b = a' - b' + 1$

2) $i = i'$ 1, $a = a'$ *or* $b = b'$

3) $i = i' + 1$, $a - b + 1 = a' + b'$

To prove 1) it suffices to check that a line $l \subset F^d$ intersectes two opposite edges of the tetrahedron $\Delta$. In the opposite case, the line $l$ intersect two faces on $\Delta$. We may assume that the points of the intersections $P, Q$ have the coordinates $(1 : p : q : 0)$ and $(0 : r : s : 1), pqrs \neq 0$.

But the line $l = (PQ)$ can't be contained into $F^d$ since the polynomial

$$f(t) = 1 + (p + rt)^d + (q + st)^d + t^d$$

isn't identically zero. This contradiction proves 1). The proof of 2) is straighforward. ∎

Thus we have 48 lines on the Fermat quartic surface $F = F^4$. They are numerated by elements of the group $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. We consider some properties of these lines.

Define an equivalence relation on the set of lines on $F$ by the formula:

$$l(i, a, b) \sim l(i', a', b') \Leftrightarrow i = i',\ a + b = a' + b', a - b = a' - b'.$$

PROPOSITION 5.2. *Each equivalence class of lines consists of two non-intersecting lines $l$ and $l'$. They compose an orbit of the normal subgroup $\{\mathrm{diag}(1, \epsilon_1, \epsilon_2, \epsilon_1 \cdot \epsilon_2), \epsilon_i^2 = 1\}$ of the group $I(m)$ acted on the set of lines. The set $l \cup l'$ is the fixed point set of an involution from the group $H \subset I(m)$ (see § 4). The group $H$ acts transitively on the set of the lines.* ∎

For example, let $a + b = 2p,\ a - b = 2q,\ p, q \in \{0, 1\} \subset \mathbb{Z}_4$. Then the lines $l(0, a, b)$, $l(0, a + 2, b + 2)$ are equivalente and they compose the fixed point set of the involution

$$\beta^{p,q} = (\sigma_1\sigma_2)^{p+1}(\sigma_1\sigma_3)^q \iota_1\iota_2\alpha_{01}\alpha_{23}.$$

COROLLARY 5.1. *Each projective line on the Fermat surface $F$ is 2-dimensional totally geodesic submanifold of the Fermat manifold $(F, m)$.* ∎

Let $Q = \{y \in \mathbb{C}P^3, y_0^2 + \ldots + y_3^2 = 0\}$ be the quadric. We identify $Q$ with $\mathbb{C}P^1 \times \mathbb{C}P^1$ such that the fibres of the projections $\pi_1, \pi_2$ onto the factors compose two families of the straight generatrices of $Q$. The mapping $q : \mathbb{C}P^3 \to \mathbb{C}P^3, (x_i) \to (x_j^2)$ map $F$ onto $Q$. The mappings $\pi_i \circ q : F \to \mathbb{C}P^1$, $i = 1, 2$ define two pencils $\mathcal{E}_1, \mathcal{E}_2$ of elliptic curves of the degree 4 in $F$. Projective lines of $F$ are mapped by $q$ onto straight generatrices of $Q$ and, hence, are contained into singular fibres of the pencils.

PROPOSITION 5.3. *The lines* $l(i,a,b)$ *and* $l(i',a',b')$ *are contained into one fibre of the elliptic pencils iff they are equivalent under the action of the normal subgroup* $\{\mathrm{diag}(1,\epsilon_1,\epsilon_2,\epsilon_3),\epsilon_i^2=1\}\subset I(m)$ *acted on the set of the lines, i.e. iff*

$$i=i',\quad a\equiv a'(\mathrm{mod}\ 2),\quad b\equiv b'(\mathrm{mod}\ 2).$$

∎

COROLLARY 5.2. *48 lines on* $F$ *are decomposed into 12 classes. Each class consists of 4 lines, which compose an singular fibre of one of the elliptic pencil. Lines from different fibres of the one pencil do not intersect.* ∎

Six singular fibres of each elliptic pencil are situated over points of the base $\mathbb{C}P^1\approx S^2$ which identify (up to a projective transformation) with two vertices $0,\infty,\pm1,\pm i$ of the octahedron (see [8], ch. I, § 11.)

PROPOSITION 5.4.

1) *The self-intersection number of a projective line on the Fermat surface* $F$ *is equal 2. Two distinct lines isn't homologous.*

2) *Let* $f_1,\dots,f_{48}$ *be the cohomology classes dual to the classes of the projective lines and* $e_0$ *is the cohomology class of the Kähler form* $m\circ J_0$. *Then*

$$e_0=\frac{1}{12}(f_1+\ldots+f_{48}).$$

3) *The* 20*-dimensional cohomology space* $H^{1,1}(F,\mathbb{C})$ *is generated by the classes* $f_1,\dots,f_{48}$.

*Proof.* 1) Since the surface $F$ has trivial canonical bundle $K_F$, the normal bundle $N_C$ of a smooth curve $C$ in $F$ coincides with the canonical bundle $K_C$, that is, the bundle of holomorphic 1-forms. Hence, the self-intersection number

$$C\cdot C=\deg N_C=-\chi(C),$$

where $\chi(C)$ is the Euler characteristic. Suppose that the curve $C$ is rational. Then $C\cdot C=-\chi(C)=-2$ and the bundle $K_C$ has only zero section. The second assertion shows that the curve $C$ is not deformable. Since the manifold $F$ is simply connected, this implies that distinct projective lines isn't cohomologous.

2) Let $M_{i,a}$ be the plane contained the point $P(i,a)\in\Delta_i\cap F$ and the opposite edge $\Delta_i'$ of the coordinate tetrahedron. The set $F\cap M_{i,a}$ consists of four projective lines. Hence, the union of these lines is linear equivalent to the class of hyperplane section which is dual to $e_0$. Averaging on 12 such planes, we prove 2).

3) Let $f_1, \ldots, f_{24}$ be the cohomology classes which are defined by the lines from singular fibers of the elliptic pencil $\mathcal{E}_1$. These classes generate the hyperplane $a^\perp \subset H^{1,1}(F, \mathbb{C})$ orthogonal to a general fibre $a$ of the pencil $\mathcal{E}_1$ [1], § 8. Let $l \subset F$ a line which is contained into a fibre of the pencil $\mathcal{E}_2$. Then $l$ isn't contained into the fibres of $\mathcal{E}_1$ and associated cohomology class $f \notin a^\perp$. Hence, the classes $f_1, \ldots, f_{24}, f$ generate the space $H^{1,1}(F, \mathbb{C})$. This proves 3). ∎

It is known that there are 27 projective lines on a cubic, in particular, on the Fermat cubic $F^3$. They generat the Picard lattice [3]. Similar result which we state without proof is true for the Fermat surface $F = F^4$.

THEOREM. *The lattice $S$ of complex algebraic cycles on the Fermat surface F is generated by the classes of 48 projective lines.* ∎

## 6. GROUP CHARACTERIZATION OF THE DISTINGUISHED COMPLEX STRUCTURES

Now we construct the involutions $\rho \in I(m)$ such that the homology class of Fix $(\rho) \subset F$ is dual to the harmonic 2-form $\epsilon_2 = \sqrt{2}\, m \circ J_1$, (see § 3,4). This implies the characterizazion of the distinguished complex structures on the Fermat manifold $(F, m)$ in terms of the stabilizers in the group $I(m)$. This results are used in § 7 to study the fixed points set of involutions.

Denote by $A \subset PGL_4(\mathbb{C})$ the finite group generated by the projective transformation

$$\sigma_{jk} = \sigma_0 \circ \sigma_i, \quad \alpha_i = \alpha_{0i} \circ \alpha_{jk}$$

where $(i, j, k)$ are cyclic permutations of (1,2,3) and $\sigma_i, \alpha_{jk}$ are defined in § 4. Then $A \approx (\mathbb{Z}_2)^4$. Let $B$ the group generated by $A$ and the complex conjugation $\tau$. We set

$$\rho_i = \alpha_{0i} \circ \alpha_{jk} \circ \tau, \quad \rho_{i+3} = \sigma_0 \sigma_i \rho_i, \quad i = 1, 2, 3.$$

The involutions $\rho_i (i = 1, \ldots, 6)$ generate the group $B \approx (\mathbb{Z}_2)^5$ and $\rho_1 \circ \rho_2 \circ \ldots \circ \rho_6 =$ id . Notice that $B$ preserves the Fermat surface $F$. Hence, we may identify $B$ with a subgroup of $I(m)$. Then $A = B \cap I^0(m)$.

THEOREM 6.1. *For each $b \in B$, the fixed points set* Fix$(b)$ *of $b$ into $F$ and the Euler characteristic* $\chi(\text{Fix}(b))$ *are described in the following table:*

**Table 2.**

| $b \in B$ | Fix $b$ | $\chi(\mathrm{Fix}\ b)$ | numbers of elements $b$ |
|---|---|---|---|
| id | $F$ | 24 | 1 |
| $A - \{\mathrm{id}\}$ | finite set | 8 | 15 |
| $\rho_j$ | connected 2 dimensional submanifold | $-8$ | 6 |
| other | $\emptyset$ | 0 | 10 |

*Proof.* It is well known [4] that for $K3$ surface $F$ $\chi(F) = 24$. Since $A \subset I^0(m)$, the second row follows from proposition 2.1. 6) To establish last row we check straighforward that the involutions $\tau$, $\sigma_{jk}\tau = (\iota_j\iota_k)\tau(\iota_j\iota_k)^{-1}, \sigma_{ij}\alpha_i\tau$ have not fixed points in $F$. Now we prove the third row.

Set

$$D = \{\mathrm{diag}(1, \delta_1, \delta_1, \delta_3), \delta_i^4 = 1\} \simeq (\mathbf{Z}_4)^3.$$

Then $D$ is a normal subgroup of $I(m)$.

LEMMA 6.1. *Let $h \in \tau G$ be an antiholomorphic involution and $Z_D(h)$ is the centralizer of $h$ into $D$. Then*

(i) *the group $Z_D(h)$ acts transitively on the set of the connected components of* Fix$(h)$. *Hence, the components have equal Euler characteristic.*

(ii) *If $\chi = \chi(\mathrm{Fix}(h)) > 0$ then* Fix$(h)$ *is the union of $\chi/2$ spheres. If $\chi = 0$, then* Fix$(h)$ *is the union of tori. If $\chi < 0$, then* Fix$(h)$ *is a connected set.*

*Proof.* First, we show that (i) implies (ii).

Let $\mathrm{Fix}(h) = C_1 \cup \ldots \cup C_k$, $C_j \cap C_j = \emptyset$, $i \neq j$ be the decomposition of Fix$(h)$ into the connected components. By (i) we have $\chi(C_1) = \ldots = \chi(C_k) = \frac{1}{k}\chi$. Since $J^h$-complex submanifolds $C_j, j = l, \ldots, k$ are oriented, (where $J^h$ is the $h$ invariant complex structure) we have $\chi(C_j) \in \{2, 0, -2, -4, \ldots\}$. Hence, if $\chi > 0$ then $X(C_j) = 2, C_j \approx S^2$. If $\chi = 0$, then $C_j \approx T^2$. Let $\chi < 0$. Note that $C_j$ are smooth $J^h$-complex curve. Using the adjunction formula for a smooth curve on a surface with the trivial canonical bundle, we have $C_j \cdot C_j = -\chi(C_j) = -\frac{1}{k}\chi > 0$. Hence, the intersection form $(C_j \cdot C_j) = \mathrm{diag}(-\frac{\chi}{k}, \ldots, -\frac{\chi}{k})$ is positively defined. On the other hand, the intersection form of complex algebraic cycles on a compact complex surface has signature $(+, -, \ldots, -)$ [3]. So $k = 1$ and $\mathrm{Fix}(h) = C_1$ is connected.

To prove (i), we define the ramified covering with the group $D$ $\pi : \mathbb{C}P^3 \to \mathbb{C}P^3$ by the formula $y_i = x_i^4 (0 \leq i \leq 3)$. It projectes the Fermat surface $F \subset \mathbb{C}P^3$ onto the plane $\mathbb{C}P^2 = \{\sum y_i = 0\}$. Since $D$ is a normal subgroup, the involution $h$ defines some involution $\bar{h} = hD = Dh$ on $\mathbb{C}P^2$. The assertion (i) is equivalent to the following statement:

(i′) The map $\pi$ projects any connected component of $\mathrm{Fix}(h)$ onto $\pi(\mathrm{Fix}(h))$.

The projection $\pi(\mathrm{Fix}(h))$ can be ramified only over points from the set $\pi(\mathrm{Fix}(h)) \cap \Delta \subset \mathrm{Fix}(\bar{h}) \cap \Delta$, where $\Delta = \{y_0 y_1 y_2 y_3 = 0\}$ is the coordinate tetrahedron. It can be checked that $\bar{h} = \tau$, $\alpha_{23} \circ \tau$ or $\alpha_{01} \circ \alpha_{23} \circ \tau$ (see § 7) and $\mathrm{Fix}(\bar{h}) \approx \mathbb{R}P^2$. We consider only the case $h = \rho_1 = \alpha_{01} \circ \alpha_{23} \circ \tau$. Then $\pi(\mathrm{Fix}(\rho_1)) = \mathrm{Fix}(\bar{\rho}_1) \approx \mathbb{R}P^2$,

$$\mathrm{Fix}(\bar{\rho}_1) \cap \Delta = \{P_1 = (1 : (-1) : 0 : 0),\ P_2 = (0 : 0 : 1 : (-1))\}.$$

This implies i′) for $h = \rho_1$. For other case, the proof is similar. Of cause we use that the pre-image of a regular point $x \in \pi(\mathrm{Fix}(h))$, $h \in \tau G$, is an orbit of $Z_D(h)$. It is easy to prove.

Now we compute $\chi(\mathrm{Fix}\,\rho_1)$. The pre-image $\pi^{-1}(P_i)$ of each ramified point $P_i$, $i = 1, 2$ consists of 4 points which are invariant by $\rho_1$. The simple calculation gives

$$Z_D(\rho_1) = \{\mathrm{diag}(1, \delta_1, \delta_2, \delta_3),\ \delta_1 = \delta_2 \cdot \delta_3\} \simeq (\mathbf{Z}_4)^2.$$

Let $v, e, f$ be the number of vertices, edges and faces of a triangulation $T$ of $\pi(\mathrm{Fix}(\rho_1)) \simeq \mathbb{R}P^2$, which has the points $P_1, P_2$ as vertices. Then $v - e + f = \chi(\mathbb{R}P^2) = 1$.

For the corresponding triangulation $T' = \pi^{-1}(T)$ of $\mathrm{Fix}(\rho_1)$ we have

$$v' = 16v - (16 - 4) \cdot 2,\quad e' = 16e,\quad f' = 16f,$$
$$\chi(\mathrm{Fix}(\rho_1)) = v' - e' + f' = -8$$

Now the lemma 6.1 (ii) shows that $\mathrm{Fix}(h)$ is a connected surface of genus 5. For the other involutions $\rho_i$, the calculations are similar. This proves the theorem. ∎

REMARK. The theorem 6.1 remain true if the Fermat surface $F$ is replaced by a quartic $X$ from the connected component $Q \ni F$ on the set of $B$-invariant smooth quartic in $\mathbb{C}P^3$. These quartic are defined by

$$q_0 \sum_{i=0}^{3} x_i^4 + cycl\ q_i(x_0^2 x_i^2 + x_j^2 x_k^2) + q_4 x_0 x_1 x_2 x_3 = 0,$$
$$q = (q_0 : q_1 : q_2 : q_3 : q_4) \in \mathbb{R}P^4.$$

PROPOSITION 6.1. *For any $B$-invariant quartic $X \in Q$, the space $H_2(X, \mathbf{C})^B$ of $B$-invariant elements from $H_2(X, \mathbf{C})$ is 1-dimensional.*

*Proof.* Set $H_i = H_i(X, \mathbf{C}), H_i^B = H_i(X, \mathbf{C})^B$. Since $H_0 = H_0^B \simeq H_4 = H_4^B \simeq \mathbf{C}$, $H_1 = H_3 = 0$, it is sufficient to prove that $k := \dim H_*^B = 3$. It well known that

$$k = |B|^{-1} \sum_{b \in B} t(b)$$

where $t(b)$ is the trace of the induced operator $b_* \in \operatorname{End} H_*(X, \mathbf{C})$.

We have

$$t(b) = tr b_*|_{H_*} tr\, b_*|_{H_0} + tr\, b_*|_{H_2} + tr\, b_*|_{H_4} =$$
$$= \sum (-1)^i tr\, b_*|_{H_i} = L(b)$$

where $L(b)$ is the Lefschetz number. By the known formula [6], $L(b) = \chi(\operatorname{Fix}(b))$. Using the results from the table 2, now we can calculate $k$ as follows:

$$k = |B|^{-1} \sum_{b \in B} t(b) = 2^{-5} \sum_{b \in B} \chi(\operatorname{Fix}(b)) =$$
$$= 2^{-5}(24 + 8 \cdot 15 - 8 \cdot 6) = 3.$$

∎

THEOREM 6.2. *Let $X$ be a quartic from $Q$ and $m$ is the Calabi-Yau metric on $X$ associated with the embedding $X \subset \mathbf{C}P^3$. Then (i) there is (unique up to a signe) $B$-invariant parallel complex structure $J$ on $X$.*

(ii) *The $J$-complex curves $C_j = \operatorname{Fix}(\rho_j)$, $j = 1, \ldots, 6$ are mutually homologous and are dual to an auto-dual parallel 2-form $\gamma$ with $\gamma^2 = 8$.*

*Proof.* (i) follows from the relation $\frac{B}{B \cap I^0(m)} \simeq \mathbf{Z}_2$ and the proposition 2.1. To prove (ii), we remark that the self-intersection number $C \cdot C$ of a smooth complex curve $C$ on $K3$-surface equals up to sign to the Euler characteristic $-\chi(C)$. In particular, for $J$-complex curves $C_j$, we have

$$C_j \cdot C_j = \chi(\operatorname{Fix}(\rho_j)) = 8.$$

By proposition 6.1., the space of $B$-invariant element $H_2(X, \mathbf{C})^B$ is 1-dimensional. Hence, it is dual to the 1-dimensional subspace of $H^2(X, \mathbf{C})$ generated by the auto-dual Kähler 2-form $m \circ J$. Since the group $B \approx (\mathbf{Z}_2)^5$ is commutative and preserves $J$, it preserves also the fixed points set $C_j = \operatorname{Fix}(\rho_j)$ and the orientation on $C_j$. This means that the homology class $[C_j]$ is $B$-invariant and, hence, it does not depend on $j = 1,\dots,6$. This proves (ii). ■

Returning to the Fermat manifold $(F, m)$, we have

COROLLARY. *Let $\tau \in I(m)$ be the involution of $F$ defined by the complex conjugation. Then there is a $\tau$-invariant integer parallel 2-form $\gamma \in \Gamma$ with $\gamma^2 = 8$. Under the notations of § 3, 4 one may assume that*

$$\gamma = \epsilon_2 = \sqrt{2} m \circ J_1 .$$

*Proof.* By theorem 6.2, there is an integer parallel $B$-invariant (and, hence, $\tau$-invariant) 2-form $\gamma$ with $\gamma^2 = 8$. By theorem 3.2, the only integer parallel 2-form with square 8 are $\pm\epsilon_1, \pm\epsilon_2$. The isometry group $I(m)$ permutes these 2-forms (see § 4). Hence, we may assume that $\gamma = \epsilon_2$. According to § 4, the associated with $\epsilon_2$ parallel complex structure is $J_1$, $\langle J_1 \rangle = \epsilon_2^{\perp}$. More precisely, we have $\epsilon_2 = \sqrt{2} m \circ J_1$. Indeed, for any parallel complex structure $J$, we have

$$(m \circ J)^2 = (m \circ J_0)^2 = \operatorname{vol} F = 4, \ \epsilon_2^2 = 8 .$$

■

Under the identification from the corollary, we have

PROPOSITION 6.2. 1) *Let $g \in G = I(m, J_0)$. Then $g\tau$ preserves the complex structure $J_t$, $t^8 = 1$, if $|g| = t^2$. In particular, $g\tau$ preserves $J_1$ if $|g| = 1$ and $g\tau$ preserves $J_\chi$ if $|g| = i$.*

2) $I(m, J_1) = I^0(m) \lambda \{1, \tau\}, I(m, J_\chi) = I^0(m) \lambda \{1, \iota_0 \tau\}$.

3) *The subgroups of $I(m)$, which preserve $J_t$ up to a signe, are*

$$I(m, \pm J_0) = I(m), \qquad I(m, \pm J_1) = H \lambda \{1, \tau\},$$
$$I(m, \pm J_\chi) = H \lambda \{1, \iota_0 \tau\}.$$

*Proof.* By results of § 4, the group $I(m) = G \cup \tau G$ acts on the space $E = \mathbf{R}\epsilon_0 + \mathbf{R}\epsilon_1 + \mathbf{R}\epsilon_2$ as the dihedral group $D_4$. More precisely, in the basis $\epsilon_0 = m \circ J_0, \epsilon_1 = -\sqrt{2} m \circ J_i, \epsilon_2 = \sqrt{2} m \circ J_1$ we have $\tau|_E = \operatorname{diag}(-1, -1, 1); g|_E = \operatorname{diag}(1, |g|)$ for $g \in G$ where $|g|$ is the matrix of the multiplication by the complex number $|g|$. Indeed, $\omega = \epsilon_1 + i\epsilon_2$ is $J_0$-holomorphic 2-form and $g^*\omega = |g|\omega$ by lemma 4.1. Now the proof is straightforward. ■

## 7. THE CLASSIFICATION OF 2-DIMENSIONAL FIXED POINTS SETS OF INVOLUTIONS

Now we enumerate connected 2-dimensional components of all sets $\mathrm{Fix}(h)$, $h \in I(m)$ on the Fermat manifold $(F, m)$. According to the proposition 2.1 and the remark at the end of § 4, we may assume that $h \in I(m) - I^0(m)$ and it is an involution. By corollary 4.2, $h$ preserve precisely one of the 5 distinguished complex structures $J_t$ (considered up to a signe). Moreover, conjugating $h$ in $I(m)$, we may suppose that $h$ preserve one of the complex structures $J_t = J_0, J_1, J_\chi$, $\chi = \frac{1}{\sqrt{2}}(1+i)$. Hence, the set $\mathrm{Fix}(h)$ is a $J_t$-complex curve. All such involutions $h$ (up to a conjugation) and its fixed points set $\mathrm{Fix}(h)$ are described in the following

THEOREM 7.1. *Any involution $g \in I(m) - I^0(m)$ is conjugated into the group $I(m)$ to one and only one of the involutions $h$ from the table 3.* ∎

There are also indicated: the $h$-invariant complex structure $J_t$, the connected components $C$ of the set $\mathrm{Fix}(h)$, the Euler characteristic $\chi(C)$ (equal up to the signe to the intersection number $C \cdot C$), the integral $d(C) = \int_C \gamma$, wher $\gamma \in \Gamma \cap \mathbf{R}^+(m \circ J_t)$ is the generator of the group $\Gamma \cap \mathbf{R}(m \circ J_t) \approx \mathbf{Z}$ (that is, $\gamma = \epsilon_0$ for $t = 0$, $\gamma = \epsilon_2$ for $t = 1$, $\gamma = \epsilon_2 - \epsilon_1$ for $t = \chi$), the number $a$ of the involutions $g$ conjugated to $h$, the number $b$ of 2-dimensional component for all sets $\mathrm{Fix}(g)$, $g = xhx^{-1}$, $x \in I(m)$. We denote by $C^\rho$ a surface of genus $\rho$.

**Table 3.**

| $h$ | $J_t$ | $\mathrm{Fix}(h)$ | $\chi(C)$ | $d(C)$ | $a$ | $b$ |
|---|---|---|---|---|---|---|
| $\sigma_0$ | $J_0$ | $C^3$ | $-4$ | 4 | 4 | 4 |
| $\alpha_{01}$ | $J_0$ | $C^3$ | $-4$ | 4 | 24 | 24 |
| $\iota_0 \iota_2 \alpha_{01} \alpha_{23}$ | $J_0$ | $C^0 + C^0$ | 2 | 1 | 24 | 48 |
| $\alpha_{01} \alpha_{23} \tau$ | $J_1$ | $C^5$ | $-8$ | 8 | 6 | 6 |
| $\sigma_0 \sigma_2 \alpha_{01} \alpha_{23} \tau$ | $J_1$ | $\emptyset$ | 0 | – | 6 | 0 |
| $\tau$ | $J_1$ | $\emptyset$ | 0 | – | 4 | 0 |
| $\iota_0^{-1} \iota_1 \tau$ | $J_1$ | $C^1$ | 0 | 4 | 12 | 12 |
| $\sigma_0 \alpha_{23} \tau$ | $J_1$ | $C^0 + C^0$ | 2 | 2 | 24 | 48 |
| $\iota_1 \tau$ | $J_x$ | $C^0$ | 2 | 4 | 16 | 16 |
| $\iota_0^{-1} \alpha_{23} \tau$ | $J_x$ | $C^2$ | $-2$ | 8 | 24 | 24 |

COROLLARY 7.1. (i) *There are 76 $J_0$-complex, 66 $J_1$-complex and 40 $J_x$-complex connected 2-dimensional submanifolds which are fixed point sets of involutions $h \in I(m)$.*

(ii) *The complete number of connected 2-dimensional submanifolds which are connected components of the sets* $\mathrm{Fix}(g)$, *is equal* $288 = 76+66+66+40+40 = 2^5 \cdot 3^2$. *Each of these submanifolds is complex with respect to one of the 5 distinguished complex structures $J_t$ (considered up to a signe).*

Outline of the proof. The first and the second columns of the table are established by the straighforward calculations based on the proposition 6.2. The Euler characteristic $\chi(\mathrm{Fix}(h))$ is calculated as in § 6 or, in some cases (for example, for $h = i_0^{-1} i_1 \tau$) by means of the analytical geometry. Then the results of the third and the fourth columns follow from the lemma 6.1. The calculation of the numbers $a$ and $b$ is straightforward. To calculate $d$, we note that for $J_0$-holomorphic involution $h$ the form $\gamma = \epsilon_0 = m \circ J_0$ is cohomologic to the Kähler form of the metric induced on $F$ from the Fubini-Study metric. Hence, $d(C)$ is equal to the degree of the algebraic curve $C \subset \mathbb{C}P^3$ or, which is the same, to the intersection number $C \cdot H$ of $C$ and a general hyperplane section $H$. For $J_1$-holomorphic involution $h$, the form $\gamma = \epsilon_2 = \sqrt{2} m \circ J_1$ is dual to the $J_1$-complex cycle $C_1 = \mathrm{Fix}(\rho_1)$ (theorem 6.2) and, hence, $d(C) = C \cdot C_1$ is equal to the number of the intersection point $|C \cap C_1|$. (Since $C$ and $C_1$ are totally geodesic submanifolds in $(F, m)$, all intersections are transversal.)

For $J_\chi$-holomorphic involution $h, \gamma = \epsilon_2 - \epsilon_1$ and $\omega_x = 2\epsilon_0 + i(\epsilon_1 + \epsilon_2)$ is the $J_\chi$-holomorphic 2-form. Hence, it integral over $J_\chi$-complex cycle $C$ vanishes. So we have

$$\begin{aligned} d(C) &= \int_C (\epsilon_2 - \epsilon_1) = +2 \int_C \epsilon_2 - \int_C (\epsilon_1 + \epsilon_2) = 2 \int_C \epsilon_2 \\ &= 2 C \cdot \mathrm{Fix}(\rho_1). \end{aligned}$$

Since the form $\epsilon_2$ can be written explicitely as the real part of the $J_0$-holomorphic 2-form $\omega$, last integral can be calculated. ∎

## 8. THE $J_1$-HOLOMORPHIC EMBEDDING OF THE FERMAT MANIFOLD INTO $\mathbb{C}P^5$ AND THE ASSOCIATED CALABI-YAU METRIC

Let $X$ be an $B$-invariant quartic in $\mathbb{C}P^3$ from the family $Q$ with the Calabi-Yau metric $m$, in particular, the Fermat manifold. (see § 6). We denote by $J_0$ (resp., $J$) the standard (resp., $B$-invariant) complex structure on $X$. Since $\tau_* J_0 = -J_0$, $\tau_* J = J$ the Kähler forms $m \circ J_0$ and $m \circ J$ are orthogonal and, hence, the structure $J_0$ and $J$ anti-commute.

Consider the fixed points sets $C_i = \mathrm{Fix}(\rho_i)$ of the involutions $\rho_i \in I(m)$, $i = 1,\ldots,6$, see § 6. By the theorem 6.1, 6.2, the sets $C_i$ are $J$-complex connected smooth curves of genus 5. The curves $C_i$ are mutually homologous and, hence, they are linear equivalent. We denote by $L$ the line bundle defined by $C_j$.

LEMMA 8.1. *Let $G$ be a smooth irreducible curve of genus $N > 0$ on a $K3$-surface $Y$ and $L = [C]$ is the associated line bundle. Then*

(i) dim $H^0(Y,O(L)) = N+1, H^1(Y,O(L)) = H^2(Y,O(L)) = 0$;

(ii) *the linear system $|C|$ has no basic points and, hence, it is defined the holomorphic map*

$$j : Y \to P(H^0(Y,O(L))^*) \simeq \mathbb{C}P^N.$$

*Proof.* Since the canonical bundle $K_Y$ is trivial, the adjunction formula implies $L|_C = K_C$ and $H^0(C,L|_C) = H^0(C,K_C) = \mathbb{C}^N$. Since $Y$ is simply connected, $H^1(Y,O) = 0$. Hence, the cohomological sequence, induced by the sequence of sheaves

$$0 \to O_Y \to O_Y(L) \to O_C(L) \to 0$$

can be written as

$$0 \to H^0(Y,O) = \mathbb{C} \to H^0(Y,O(L)) \to \mathbb{C}^N =$$
$$= H^0(C,O(K_C)) \to 0.$$

So we have $H^0(Y,O(L)) \simeq \mathbb{C}^{N+1}$. By the duality of Kodaira-Serre, $H^2(Y,O(L)) = H^0(Y,O(L^*))^* = 0$, since the divisor $C$ is effective. Using the addivity of Euler characteristic, we have

$$\begin{aligned}\chi(Y,O(L)) &= h^0(Y,L) - h^1(Y,L) + h^2(Y,L) = \\ &= N+1-h^1(Y,L) = \chi(Y,O) + \chi(C,O(K_C)) = \\ &= 2+(N-1) = N+1.\end{aligned}$$

Hence, $H^1(Y,O(L)) = 0$. These prove (i) The surjectivity of the map $H^0(Y,O(L)) \to H^0(C,O(L|_C))$ implies that the basical set of $|C|$ is contained into the set of points where all sections of the bundle $L|_C = K_C$ vanish. The last set is emply. This proves the lemma.

Applying the lemma to the $J$-holomorphic curves $C_i$ of genus 5 on $X \in Q$, we receive a $J$-holomorphic map

$$j : X \to \mathbb{C}P^5 = P(H^0(X,O(L))^*), \quad L = [C_j].$$

■

PROPOSITION 8.1. *All fibers of the map $j : X \to \mathbb{C}P^5$ are finite and the image $j(X)$ is contained into the intersection of three quadric in $\mathbb{C}P^5$.*

*Proof.* By theorem 6.2., the curve $C_i (i = 1, \dots, 6)$ is dual to the cohomology class $\sqrt{2}[m \circ J]$. Hence, the line bundle $L = [C_i]$ is positive and, by Kodaira theorem, it is ample. This implies that all fibers of $j$ is finite. To prove the second assertion, we compute $h^0(X, L \otimes L)$. As in the proof of the lemma 8.1, we show that $h^0(X, L \otimes L) = \chi(X, O(L \otimes L))$. Then the Noether formula gives

$$\begin{aligned} h^0(X, L \otimes L) &= \chi(X, O) + \frac{1}{2}((L \otimes L) \cdot (L \otimes L) + (L \otimes L) \cdot K) \\ &= 2 + \frac{1}{2}(2C_i) \cdot (2C_i) = \\ &= 2 + 2 \cdot 8 = 18. \end{aligned}$$

On the other hand, $h^0(CP^5, O(2)) = 21 = 18 + 3$. Now the arguments from [3], ch. 4, § 5 establishe the assertion. ∎

We state without proof more precise result.

THEOREM 8.1. *The divisors $C_i = \mathrm{Fix}(\rho_i)$ $(i = 1, \dots, 6)$ on the complex surface $(X, J)$ are very ample, that is the associated holomorphic map $j : X \to \mathbb{C}P^5 = P(H^0(X, O(L))^*)$ is an embedding. The Calabi-Yau metric $m'$ associated with embedding $j$ is related with the standard Calabi-Yau metric $m$ on $X$ (associated with the embedding $X \subset \mathbb{C}P^3$) by: $m' = \sqrt{2}m$. The image $j(X)$ coincides with the smooth complete intersection of three diagonal quadric of the form $\{\xi \in \mathbb{C}P^5, \ a_1\xi_1^2 + \dots + a_6\xi_6^2 = 0\}$. The group $B \subset I(m)$ acts in $j(X)$ as the projective group*

$$\{\mathrm{diag}(1, \epsilon_1, \dots, \epsilon_5), \epsilon_1 = \pm 1\} \simeq (\mathbf{Z}_2)^5.$$

∎

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